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A Method for Analyzing Unreplicated Factorial Experiments

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ABSTRACT The analysis of unreplicated two-level factorial and fractional factorial experiments presents a challenge because after estimation of the effects by contrasts, there are no degrees of freedom left to estimate the error variance. Consequently, standard t tests cannot be used to identify the "active" effects. In practice, the standard method for identifying active effects continues to be a probability plot of the contrasts. There is a subjective element in deciding what constitutes an "active" effect in using the probability plot. This subjective element has motivated the recent flurry of activity to provide objective methods. Presently we propose yet another method based on hypothesis testing. The ability of the proposed method to detect "active" effects is found to be satisfactory.

ABSTRAK Analisis ujikaji faktoran dua-aras tanpa replikasi dan ujikaji faktoran pecahan merupakan suatu cabaran kerana setelah kesan dianggar oleh kontras, tiada darjah kebebasan tertinggal untuk menganggar varians ralat. Jadi,ujian t yang piawai tidak boleh digunakan untuk mencamkan kesan yang "aktif". Pada praktisnya, kaedah piawai untuk mencamkan kesan aktif masih diberi oleh plot kebarangkalian bagi kontras. Terdapat unsur subjektif dalam penentuan kesan "aktif" dengan menggunakan plot kebarangkalian. Unsur subjektif ini telah menjadi daya yang membangkitkan berbagai aktiviti untuk mencari kaedah objektif. Kini kami mencadangkan satu lagi kaedah yang berdasarkan ujian hipotesis. Didapati bahawa kesanggupan kaedah yang dicadangkan untuk mencamkan kesan "aktif" adalah memuaskan.

(Factorial design, fractional replication, contrasts, active effects)

INTRODUCTION

In an unreplicated two-level factorial design or fractional factorial design, the observation taken when the levels of the k factors are respectively $i, j, \ldots, K$ may be denoted as $y_{ij \ldots K}$. A linear combination of the $y_{ij \ldots K}$, based on the coefficients $h_{ij \ldots K}$, may be written as

$$c = \sum h_{ij \ldots K} \, y_{ij \ldots K}$$

where the summation is over all the relevant combinations of the levels of the k factors, and the sum of all the coefficients is zero.

Two contrasts c_1 and c_2 are said to be orthogonal if the sum of the products of their corresponding coefficients $h^{(1)}_{ij \ldots K}$ and $h^{(2)}_{ij \ldots K}$ is zero, i.e

$$\sum h^{(1)}_{ij \ldots K} \, h^{(2)}_{ij \ldots K} = 0$$

Suppose a total of n distinct orthogonal contrasts $c_1, c_2, \ldots, c_n$ can be formed. The contrasts are then measures of the effects of the factors which can be classified as main effect, two-factor interaction effect or other higher order interaction effect.

If all the effects are inert and the observations have a normal distribution with a common variance σ^2, then $c_1, c_2, \ldots, c_n$ can be treated as a collection of n independent and identical variates with mean 0 and a common variance.

This property of $c_1, c_2, \ldots, c_n$ forms the basis for detecting the active effects.

The standard analysis procedure for an unreplicated two-level factorial design is the normal (or half-normal) plot of the estimated factor effects (see [1]). However, unreplicated designs are so widely used in practice that many formal analysis procedures have been proposed to overcome the subjectivity of the normal probability plot. (see for example, Seheult and Tukey [2], Box and Meyer [3], Johnson and Tukey [4], Voss [5], Lenth [6], Benski [7], Bissell [8, 9], Berk and Picard [10], Juan and Pena [11], Loh [12], Le and Zamar [13], Dong [14], Schneider, Kasperski and Weissfeld [15] and Venter and Steel [16].

Hamada and Balakrishnan [17] compared some of these methods. They found that the method proposed by Lenth [6] has good power to detect significant effects. It is also easy to implement, and as a result, it is beginning to appear in some software packages for analysing data from unreplicated factorials.

Presently we suggest an alternative method. The proposed method is found to have approximately the desired small probability of declaring the 'extreme' effects to be active when actually all the effects are inert. When the contrasts in the model with only inert effects are arranged in an ascending order to $c_{(1)}, c_{(2)}, \ldots, c_{(n)}$ and the last k_2 values $c_{(n-k_2+1)}, c_{(n-k_2+2)}, \ldots, c_{(n)}$ are changed artificially to some "unreasonably large" values, the alternative method is found to have a larger probability of declaring the effects which correspond to the resulting values $c_{(n-k_2+1)}, c_{(n-k_2+2)}, \ldots, c_{(n)}$ to be active effects when compared with Lenth's method.

AN ALTERNATIVE METHOD

Let c_i be the estimate of the ith effect (ith contrast), $i = 1, 2, \ldots, n$ in the unreplicated factorial experiment. When there are no "active" effects, $c_1, c_2, \ldots, c_n$ may be considered to be a set of n independent random variables with mean 0 and variance σ^2.

Suppose we order $c_1, c_2, \ldots, c_n$ in an ascending order to $c_{(1)}, c_{(2)}, \ldots, c_{(n)}$ and obtain the normal probability plot. If $c_{(1)}, c_{(2)}, \ldots, c_{(k_1)}$ and $c_{(n-k_2+1)}, c_{(n-k_2+2)}, \ldots, c_{(n)}$ appear to fall off the straight line in the plot, then we may estimate the variance σ^2 based on $c_{(k_1+1)}, c_{(k_1+2)}, \ldots, c_{(n-k_2)}$. Based on the estimate of σ^2, we would be able to judge whether $c_{(k_1+1)} - c_{(k_1)}$ and $c_{(n-k_2+1)} - c_{(n-k_2)}$ are unexpectedly large or not. The above idea forms the basis for the following alternative method for detecting "active" effects.

We first generate N values of $(c_1, c_2, \ldots, c_n)$ of which $c_i \sim N(0, \sigma^2)$, $i = 1, 2, \ldots, n$. For each generated value of $(c_1, c_2, \ldots, c_n)$, we find

$$c_{(1)}, c_{(2)}, \ldots, c_{(n)} \qquad (1)$$

$$d_{k_2}^{(2)} = c_{(n-k_2+1)} - c_{(n-k_2)} \qquad (2)$$

$$\hat{\gamma}_{k_1,k_2}^2 = \frac{1}{n-(k_1+k_2)-1} \sum_{i=k_1+1}^{n-k_2} (c_{(i)} - m_k)^2 \qquad (3)$$

where m_k is the median of $c_{(k+1)}, c_{(k+2)}, \ldots, c_{(n-k)}$ and $k = \text{maximum}(k_1, k_2)$.

We next find the average value $\bar{\gamma}_{k_1,k_2}^2$ of the N values of $\hat{\gamma}_{k_1,k_2}^2$, and the value $d_{k_2}^{(2)*}$ such that among the N values of $d_{k_2}^{(2)}, [N\alpha]$ ($[x]$ is the largest integer $\le x$) values will be larger than $d_{k_2}^{(2)*}$.

Then, for each selected value of (n, k_1, k_2, α), we tabulate the values of $\sigma, \bar{\gamma}_{k_1,k_2}^2$ and $d_{k_2}^{(2)*}$. Some of the tabulated results are shown in Table 1.

Table 1. Values of $\sigma, \bar{\gamma}^2_{k_1,k_2}$ and $d^{(2)*}_{k_2}$ when $n = 18, k_1 = k_2 = 1, \alpha = 0.01$, $N = 10000$

σ	$\hat{\gamma}^2_{k_1,k_2}$	$d^{(2)*}_{k_2}$
1.00E-04	6.84E-09	1.85E-04
2.00E-04	2.74E-08	3.60E-04
3.00E-04	6.14E-08	5.41E-04
4.00E-04	1.09E-07	7.40E-04
5.00E-04	1.70E-07	9.06E-04
6.00E-04	2.46E-07	0.001105577
7.00E-04	3.34E-07	0.001307324
8.00E-04	4.35E-07	0.001456988
9.00E-04	5.50E-07	0.001640424
0.01	6.80E-05	0.018608715
0.02	2.72E-04	0.036614271
0.03	6.13E-04	0.055323068
0.04	0.001088466	0.074473785
0.05	0.001698023	0.091818644
0.06	0.002456978	0.109677442
0.07	0.003353211	0.129296721
0.08	0.004371582	0.147229086
0.09	0.005522766	0.168639028
0.10	0.006852215	0.183105952
…	…	…
14.99	153.3660624	27.156018201
15.00	153.2114758	27.293726599

We may now declare $c_{(n-k_2+1)}, c_{(n-k_2+2)}, \ldots, c_{(n)}$ to be "active" effects if

$$c_{(n-k_2+1)} - c_{(n-k_2)} > d_{k_2}^{(2)*}$$

where $d_{k_2}^{(2)*}$ is found from the table for $\sigma, \bar{\gamma}_{k_1,k_2}^2$, $d_{k_2}^{(2)*}$ at the row of which the value of $\bar{\gamma}_{k_1,k_2}^2$ is approximately equal to that of $\hat{\gamma}_{k_1,k_2}^2$ based on the observed $(c_1, c_2, \cdots, c_n)$.

To find out whether $c_{(1)}, c_{(2)}, \ldots, c_{(k_1)}$ are "active" effects, we replace $d_{k_2}^{(2)}$ and $d_{k_2}^{(2)*}$ by $d_{k_1}^{(1)} = c_{(k_1+1)} - c_{(k_1)}$ and $d_{k_1}^{(1)*}$ respectively and repeat the same procedure.

PERFORMANCE OF THE ALTERNATIVE METHOD

To investigate the performance of the above method of detecting "active" effects, we perform the following simulation.

Suppose (n, σ, k_1, k_2) is chosen. We generate N values of $(c_1, c_2, \ldots, c_n)$ where $c_i \sim N(0, \sigma^2)$, $i = 1, 2, \ldots, n$. For each generated value of $(c_1, c_2, \ldots, c_n)$, we compute $(c_{(1)}, c_{(2)}, \ldots, c_{(n)})$, $d_{k_2}^{(2)}$ and $\hat{\gamma}_{k_1,k_2}^2$ (see the Equations (2) and (3)).

From the table for σ, $\bar{\gamma}_{k_1,k_2}^2$, and $d_{k_2}^{(2)*}$ which corresponds to the chosen values of *n*, k_1 and k_2, we find two values ($\bar{\gamma}_{k_1,k_2}^{(1)\,2}$ and $\bar{\gamma}_{k_1,k_2}^{(2)\,2}$ say) such that

$$\bar{\gamma}_{k_1,k_2}^{(1)\,2} \le \hat{\gamma}_{k_1,k_2}^2 \le \bar{\gamma}_{k_1,k_2}^{(2)\,2}$$

and note the two values of $d_{k_2}^{(2)*}$ which correspond to $\bar{\gamma}_{k_1,k_2}^{(1)\,2}$ and $\bar{\gamma}_{k_1,k_2}^{(2)\,2}$. We next use interpolation to find the approximate value of $d_{k_2}^{(2)*}$ which corresponds to $\hat{\gamma}_{k_1,k_2}^2$. The effects $c_{(n-k_2+1)}$, $c_{(n-k_2+2)}$, $\ldots$, $c_{(n)}$ are declared to be "active" if

$$c_{(n-k_2+1)} - c_{(n-k_2)} > d_{k_2}^{(2)*}. \tag{4}$$

We now find, among the N values of $\underset{\sim}{c} = (c_1, c_2, \ldots, c_n)^T$, the number N^* of $\underset{\sim}{c}$, for which (4) is satisfied. The probability of declaring $c_{(n-k_2+1)}$, $c_{(n-k_2+2)}$, $\cdots$, $c_{(n)}$, to be "active" effects is then estimated by $P_{k_2}^{(2)} = N^*/N$.

Tables 2 – 5 show some of the values of $P_{k_2}^{(2)}$. These tables show that, the difference

δ = [(target probability $1-\alpha$) – (estimated probability $P_{k_2}^{(2)}$)] × 100%

is always positive, and in most cases $\delta < 2.5\%$.

We next compare the alternative method with Lenth's method which is described below:

(1) Compute PSE = 1.5. $median_{\{|c_i|<2.5s_0\}} \, |c_i|$, where $s_0 = 1.5.median\,|c_i|$.

(2) Compute ME = $t_{0.975;df}$ PSE, where $t_{0.975;df}$ is the 97.5% point of a t distribution with degrees of freedom $df = n/3$.

(3) The *i*th effect is declared active if $c_i > ME$.

To investigate the ability of the alternative method and Lenth's method to detect "active" effects, we first generate N values of **c**= $(c_1, c_2, \ldots, c_n)^T$ where $c_i \sim N(0, \sigma^2)$, $i = 1, 2, \ldots, n$. For each generated value of **c**, we change the last k_2 values $c_{(n-k_2+1)}, c_{(n-k_2+2)}, \ldots, c_{(n)}$ in $c_{(1)}, c_{(2)}, \ldots, c_{(n)}$ to $f\sigma, \left(f+\frac{1}{2}\right)\sigma, \ldots, \left(f+\frac{k_2-1}{2}\right)\sigma$ where $f \ge 3$, and find the estimate $P_{k_2}^{(2)}$ of the probability that the effects which correspond to $c_{(n-k_2+1)}, c_{(n-k_2+2)}, \ldots, c_{(n)}$ are declared to be active effects. Some results for $P_{k_2}^{(2)}$ are shown in Figures 1-4.

Figures 1-4 show that when $f = 4$, the value of $P_{k_2}^{(2)}$ for the alternative method is above 0.8, and when $f = 4.5$, the value of $P_{k_2}^{(2)}$ is very close to 1. Furthermore the values of $P_{k_2}^{(2)}$ for the alternative method are larger than those for Lenth's method.

Table 2. The estimated probability of declaring $c_{(n-k_2+1)}, c_{(n-k_2+2)}, \cdots, c_{(n)}$ to be "active" effects under all inert effects when $(n, k_1, k_2, \alpha) = (18, 1, 1, 0.01)$

σ	$P_{k_2}^{(2)}$		σ	$P_{k_2}^{(2)}$		σ	$P_{k_2}^{(2)}$
0.1	0.9764		4.0	0.9774		8.0	0.9798
1.0	0.9777		5.0	0.9792		9.0	0.9759
2.0	0.9770		6.0	0.9784		10.0	0.9766
3.0	0.9798		7.0	0.9792		10.5	0.9769

Table 3. The estimated probability of declaring $c_{(n-k_2+1)}, c_{(n-k_2+2)}, \cdots, c_{(n)}$ to be "active" effects under all inert effects when $(n, k_1, k_2, \alpha) = (18, 2, 2, 0.01)$

σ	$P_{k_2}^{(2)}$		σ	$P_{k_2}^{(2)}$		σ	$P_{k_2}^{(2)}$
0.1	0.9784		4.0	0.9749		8.0	0.9736
1.0	0.9788		5.0	0.9776		9.0	0.9764
2.0	0.9760		6.0	0.9778		10.0	0.9758
3.0	0.9752		7.0	0.9759		10.5	0.9753

Table 4. The estimated probability of declaring $c_{(n-k_2+1)}, c_{(n-k_2+2)}, \cdots, c_{(n)}$ to be "active" effects under all inert effects when $(n, k_1, k_2, \alpha) = (18, 3, 3, 0.01)$

σ	$P_{k_2}^{(2)}$		σ	$P_{k_2}^{(2)}$		σ	$P_{k_2}^{(2)}$
0.1	0.9742		4.0	0.9711		8.0	0.9747
1.0	0.9764		5.0	0.9724		9.0	0.9728
2.0	0.9755		6.0	0.9755		10.0	0.9714
3.0	0.9748		7.0	0.9742		10.5	0.9715

Table 5. The estimated probability of declaring $c_{(n-k_2+1)}$, $c_{(n-k_2+2)}$, $\cdots$, $c_{(n)}$ to be "active" effects under all inert effects when $(n, k_1, k_2, \alpha) = (18, 4, 4, 0.01)$

σ	$P_{k_2}^{(2)}$		σ	$P_{k_2}^{(2)}$		σ	$P_{k_2}^{(2)}$
0.1	0.9686		4.0	0.9690		8.0	0.9693
1.0	0.9678		5.0	0.9718		9.0	0.9698
2.0	0.9662		6.0	0.9719		10.0	0.9666
3.0	0.9681		7.0	0.9702		10.5	0.9702

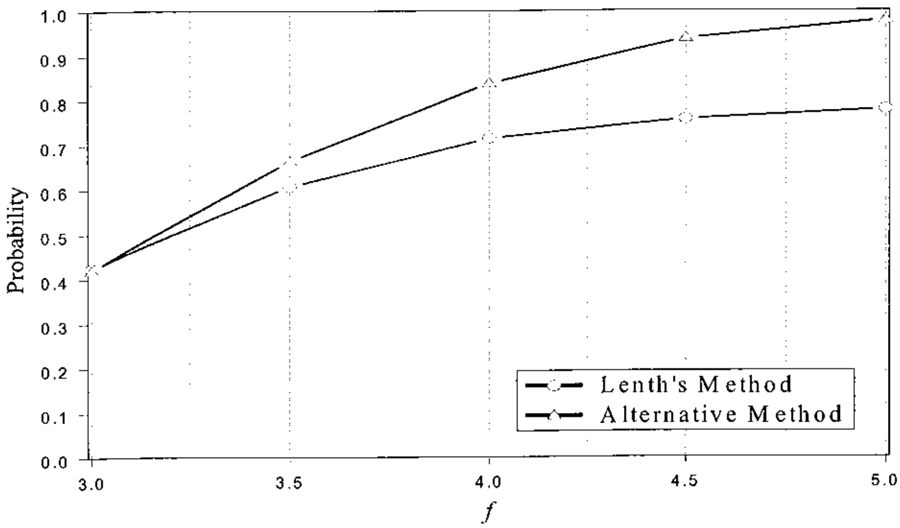

Figure 1. The probability that the effects which correspond to $c_{(n-k_2+1)}$, $c_{(n-k_2+2)}$, $\cdots$, $c_{(n)}$ are declared to be active effects when $k_2 = 1$, $\alpha = 0.05, n = 15$

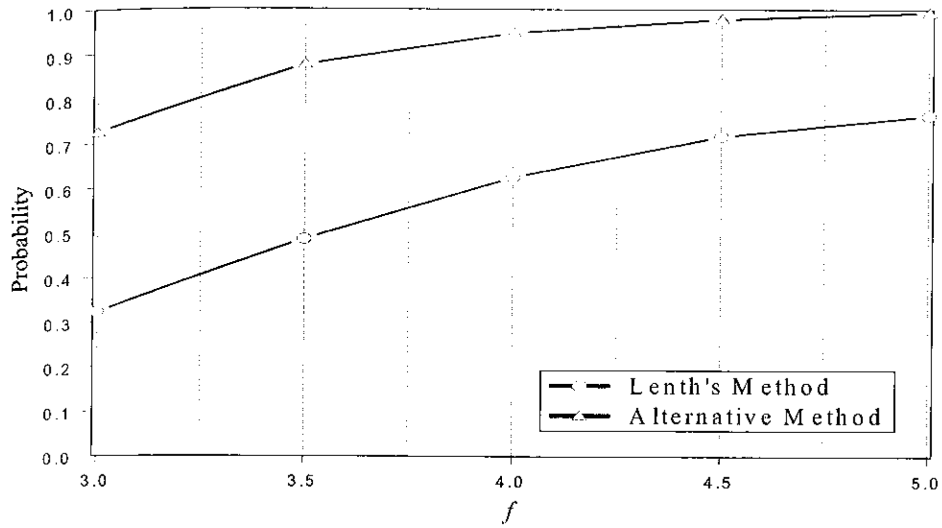

Figure 2. The probability that the effects which correspond to $c_{(n-k_2+1)}, c_{(n-k_2+2)}, \cdots, c_{(n)}$ are declared to be active effects when $k_2 = 2$, $\alpha = 0.05, n = 15$.

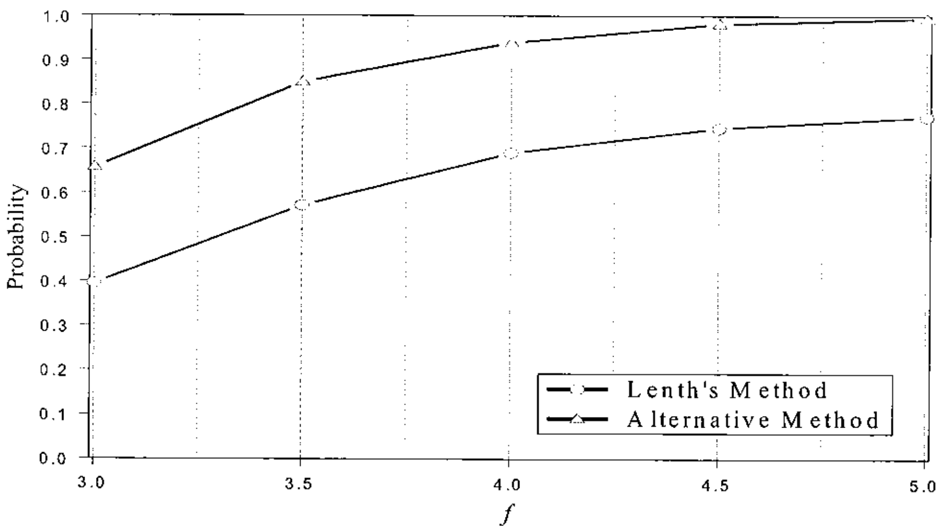

Figure 3. The probability that the effects which correspond to $c_{(n-k_2+1)}, c_{(n-k_2+2)}, \cdots, c_{(n)}$ are declared to be active effects when $k_2 = 3$, $\alpha = 0.05, n = 15$.

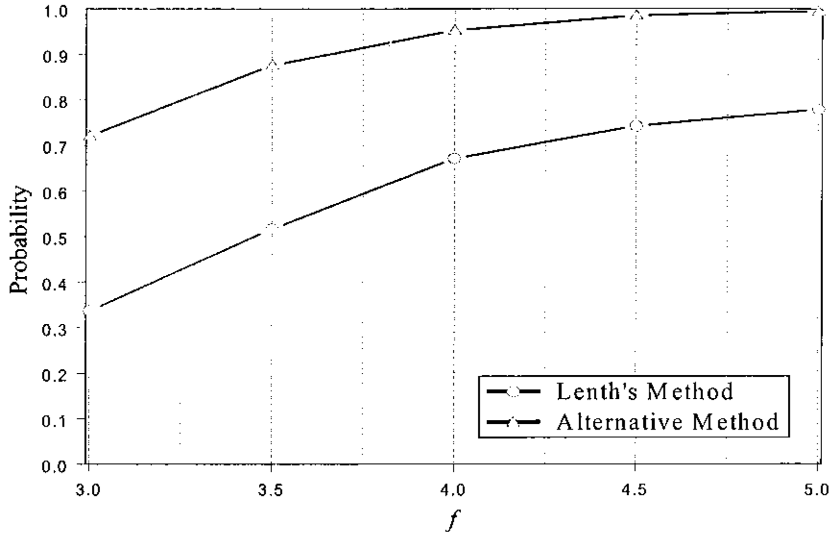

Figure 4. The probability that the effects which correspond to $c_{(n-k_2+1)}, c_{(n-k_2+2)}, \ldots, c_{(n)}$ are declared to be active effects when $k_2 = 4$, $\alpha = 0.05, n = 15$.

CONCLUDING REMARKS

The ideas contained in the alternative method may be used to form similar procedures for detecting "active" effects when the observations have a non-normal distribution. In the non-normal situations it is expected that apart from estimating the variance σ^2, we also need to estimate the skewness are kurtosis. Furthermore, it is also anticipated that the value of n needs to be fairly large for the resulting procedure to be effective.

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